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THE JOHNSON GRAPH $J(d, r)$ IS UNIQUE IF $(d, r) \neq (2, 8)$

Paul TERWILLIGER
Department of Mathematics, Ohio State University, Columbus, OH 43210, U.S.A.

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We use the classical Root Systems to show the Johnson graph $J(d, r)$ $(2 \leqslant 2d \leqslant r < \infty)$ is the unique distance-regular graph with its intersection numbers when $(d, r) \neq (2, 8)$. Since this exceptional case has been dealt with by Chang [6] this completes the characterization problem for the Johnson graph.

Introduction

Let Γ denote a finite, connected, undirected graph, with a vertex set which we also denote by Γ, and an edge set $E\Gamma$, a set of two element subsets of Γ. A walk of length n connecting vertices u and v in Γ is a sequence $\{u = v_0, v_1, \ldots, v_n = v\}$, $v_i \in \Gamma$ $(0 \leqslant i \leqslant n)$, where $(v_i, v_{i+1}) \in E\Gamma$ $(0 \leqslant i \leqslant n-1)$. The distance $\partial(u, v)$ between vertices u and v is the length of the shortest walk connecting them. The diameter d of Γ is the maximum value ∂ takes on.

For any positive integer r set $\Omega_r = \{1, 2, \ldots, r\}$. The Johnson graph $J(d, r)$ $(2 \leqslant 2d \leqslant r)$ (also called the graph of type T_d or triangular type) is the graph whose vertex set consists of all subsets of Ω_r of order d, with vertices adjacent if their intersection has order $d-1$. We note $J(d, r)$ has diameter d, and has the distance-regularity property: for any integers i, j, l $(0 \leqslant l \leqslant d)$ and any vertices u and v with $\partial(u, v) = l$, the intersection number s_{ijl} of vertices a distance i from u and a distance j from v depends only on i, j, and l, not on u and v. Setting c_i, a_i, and b_i equal to $s_{i-1,1,i}$, $s_{i,1,i}$, and $s_{i+1,1,i}$ for all i $(0 \leqslant i \leqslant d)$, one can readily verify (or see [3]) that for the Johnson graph we have

$$c_i = i^2, \quad a_i = i(r-2i), \quad b_i = (d-i)(r-d-i) \qquad (0 \leqslant i \leqslant d).$$

We note (see Biggs [2]) that for any distance-regular graph all intersection numbers are determined from c_i, a_i, and b_i $(0 \leqslant i \leqslant d)$.

Many authors have asked whether $J(d, r)$ is the unique distance-regular graph with its own intersection numbers, and there are some partial results.

The uniqueness of $J(2, r)$ was proved by Shrikhande [19] for $r < 6$, Hoffman [10] and Chang [6] for $r = 7$, and Conner [7] for $r > 8$. Chang showed there were exactly 4 nonisomorphic graphs with the intersection numbers of $J(2, 8)$. The graph $J(3, r)$ was shown to be unique by Aigner [1] for $r \leqslant 8$, Moon [14] for $r = 9$

or 10, Liebler [12] for $11 \leqslant r \leqslant 16$, Rolland [18] for $r \geqslant 9$, and Bose and Lasker [4] for $r > 16$. Dowling [8] proved $J(d, r)$ was unique for $r > 2d(d-1)+4$. This condition was weakened to $r = 2d+1$, $2d+2$, $3d$, $3d+1$, or $r \geqslant 4d$ by Moon in [13], [14], and [15], and weakened further to $r \geqslant 20$ in [16]. In this paper we finish the problem by settling the remaining cases. Specifically we prove the following

Theorem 1.1. *Let Γ be any distance-regular graph with diameter d ($1 \leqslant d < \infty$) and intersection numbers*

$$c_i = i^2, \tag{1.1}$$

$$a_i = i(r-2i), \qquad (0 \leqslant i \leqslant d) \tag{1.2}$$

$$b_i = (d-i)(r-d-i), \tag{1.3}$$

for some integer r ($2d \leqslant r$). If $(d, r) = (4, 8)$ or $r \neq 7, 8, 9$, then $\Gamma \simeq J(d, r)$.

This and our introductory remarks imply the following result.

Corollary 1.2. *The Johnson graph $J(d, r)$ ($2 \leqslant 2d \leqslant r$) is the unique distance-regular graph with its own intersection numbers if $(d, r) \neq (2, 8)$.*

We remark that the above mentioned characterizations of $J(d, r)$ by Aigner, Bose and Laskar, and Dowling are more general than ours. They do not assume the graph is distance-regular, only that it has $\binom{r}{d}$ vertices, and that $b_0 = d(r-d)$, $a_1 = r-2$, and $c_2 = 4$ (Dowling assumed $c_2 \leqslant 4$). It is apparently an open question whether these assumptions suffice to prove the uniqueness of $J(d, r)$ for all (d, r), $(d, r) \neq (2, 8)$.

Our proof of Theorem 1.1 runs as follows.

In the spirit of Cameron, Goethels, Seidel, and Shult [5], we show the set of edges of any distance-regular graph satisfying (1.1)–(1.3) can be interpreted as an irreducible set Δ of equi-length vectors in some Euclidean space, with angles at 0°, 60°, 90°, 120°, and 180°; in short as a subset of one the classical root systems of type A_n, D_n, E_6, E_7, or E_8. We then show for $r \neq 7, 8, 9$, Δ must be contained in a root system of type A_n, which enables us to prove the original graph Γ was a Johnson graph. If $(d, r) = (4, 8)$, we use a special argument.

Definition 1.3. To keep our notation simple, we fix once and for all integers d and r, which we assume are in the range $5 \leqslant r$ and $2 \leqslant d \leqslant r/2$ in order to avoid trivialities. By Γ^* we shall mean the graph $J(d, r)$. We pick any distance-regular graph satisfying (1.1)–(1.3) and denote it by Γ. Our goal is to prove $\Gamma \simeq \Gamma^*$. We use the convention that for any variable v referring to Γ, the variable v^* refers to Γ^*.

We now fix some notation. E will refer to a real Euclidean space with inner product $\langle , \rangle$. $\mathbb{R}^n$ is the Euclidean space of all real n-tuples, with the standard inner product. $B^n = \{e_1, \ldots, e_n\}$ will refer to the standard orthonormal basis for $\mathbb{R}^n$. We write

$$\delta_n = e_1 + e_2 + \cdots + e_n.$$

We will be dealing with the classical root systems, all of whose roots are the same length, which we can take to be $\sqrt{2}$. Following Hiller [9] we call them simply laced root systems. They satisfy the following definition.

Definition 1.4. A subset Φ of a Euclidean space E is called a simply laced root system if:

(i) Φ is finite and spans E; (1.4)

(ii) $\langle s, s \rangle = 2, \quad s \in \Phi$; (1.5)

(iii) $\langle s, t \rangle \in \{0, \mp 1, \mp 2\}, \quad s, t \in \Phi$; (1.6)

(iv) $s - \langle s, t \rangle t$ is in Φ for all $s, t \in \Phi$. (1.7)

Definition 1.5. A set X of vectors in a Euclidean space E is *irreducible* if it cannot be partitioned into the union of two proper subsets so that each vector in one set is orthogonal to each vector in the other.

We now give some examples of irreducible root systems. (See Humphreys [11, p. 63].)

Example 1.6. Let E be the n-dimensional subspace of $\mathbb{R}^{n+1}$ orthogonal to δ_{n+1}. Then the root system A_n $(1 \leq n)$ is the following set of vectors in E.

$$A_n = \{e_i - e_j \mid i, j \in \Omega_{n+1}, i \neq j, e_i, e_j \in B^{n+1}\}. \tag{1.8}$$

Now set $E = \mathbb{R}^n$. Then the root system D_n $(4 \leq n)$ is the set

$$D_n = \{\mp e_i \mp e_j \mid i, j \in \Omega_n, i \neq j, e_i, e_j \in B^n\}. \tag{1.9}$$

Let $E = \mathbb{R}^8$. The root system E_8 consists of D_8 and all vectors of the form

$$\tfrac{1}{2} \sum_{i=1}^{8} \alpha_i e_i \quad \left(\alpha_i = \mp 1, \prod_{i=1}^{8} \alpha_i = 1, i \in \Omega_8, e_i \in B^8\right). \tag{1.10}$$

The root system E_7 is the subset of E_8 orthogonal to any one of its roots. E_7 can also be represented as follows: Let E be the 7 dimensional subspace of $\mathbb{R}^8$ consisting of vectors orthogonal to δ_8. Then $E_7 \subseteq E$ consists of A_7 and all vectors of the form

$$\left(\sum_{i \in P} e_i\right) - \tfrac{1}{2}\delta_8 \quad (P \text{ a size 4 subset of } \Omega_8, e_i \in B^8). \tag{1.11}$$

Finally if s and t are any roots in E_8 with $\langle s, t\rangle = 1$ we can represent E_6 by the subset of E_8 orthogonal to s and t.

As the next lemma shows, this is the complete list.

Lemma 1.7 [11, p. 57]. *Let Φ be any simply laced root system. Then Φ decomposes uniquely as the orthogonal union of irreducible root systems, each of type A_n, D_n, E_6, E_7, or E_8.*

The next lemma shows we can always embed certain sets of vectors in root systems.

Lemma 1.8. *Let Δ be a finite set of vectors that span a Euclidean space E, and suppose:*

(i) $\langle s, s\rangle = 2,\quad s \in \Delta$;

(ii) $\langle s, t\rangle \in \{0, \mp 1, \pm 2\}$ for any $s, t \in \Delta$.

Then Δ is contained in a simply laced root system Φ. If Δ is reducible, so is Φ.

Proof. Partition Δ into an orthogonal union $\Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_h$ of irreducible subsets. Let Φ_i $(1 \leqslant i \leqslant h)$ be the set of vectors in E of square length 2 that can be expressed as integral linear combinations of vectors in Δ_i, and set $\Phi = \bigcup_{i=1}^{h} \Phi_i$. Then Φ satisfies (1.5).

If s and t are any vectors in Φ, the construction of Φ and (ii) makes their inner product an integer. In fact from the Cauchy-Schwartz inequality we must have

$$\langle s, t\rangle^2 \leqslant \langle s, s\rangle \langle t, t\rangle = 4$$

forcing

$$\langle s, t\rangle \in \{0, \mp 1, \mp 2\}.$$

Thus (1.6) is satisfied. To prove (1.7), note that for all $s, t \in \Phi$, $s - \langle s, t\rangle t$ is just s, and hence in Φ, unless s and t are in the same component Φ_i $(1 \leqslant i \leqslant h)$ of Φ, and in this case one checks $s - \langle s, t\rangle t$ has square length 2 and is an integral linear combination of roots in Δ_i, putting it in $\Phi_i \subseteq \Phi$. Thus Φ satisfies (1.7). Finally since Φ can be viewed as a set of points on the surface of a finite dimensional sphere in E of radius $\sqrt{2}$, that are mutually a distance $\geqslant \sqrt{2}$ apart, the finiteness of Φ follows from the compactness of the sphere. Thus (1.4) is satisfied and Φ is a simply laced root system. □

2. A representation of $J(d, r)$

We now compare some algebraic properties of Γ and Γ^*. Let $A_0 = I$, $A_1, \ldots, A_d$ be symmetric matrices, with rows and columns indexed by the vertices of Γ, such that for all $u, v \in \Gamma$ we have

$$(A_i)_{uv} = \begin{cases} 1 & \text{if } \partial(u, v) = i \\ 0 & \text{else.} \end{cases} \quad (0 \leq i \leq d)$$

We call $A = A_1$ the *adjacency matrix* for Γ. From the distance-regularity of Γ (see Biggs [2, p. 136]) we have

$$AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \quad (1 \leq i \leq d-1) \tag{2.1}$$

and

$$AA_d = b_{d-1}A_{d-1} + a_dA_d, \tag{2.2}$$

where c_i, a_i, and b_i $(0 \leq i \leq d)$ are given in (1.1)–(1.3).

The *adjacency algebra* $\mathscr{A}(\Gamma)$ is the algebra of all matrices with complex coefficients that can be expressed as polynomials in A. The equations (2.1) and (2.2) tell us the matrices A_i $(0 \leq i \leq d)$ are polynomials in A of degree i and hence are in $\mathscr{A}(\Gamma)$.

Lemma 2.1. *The map $A^* \to A$ induces an algebra isomorphism between $\mathscr{A}(\Gamma^*)$ and $\mathscr{A}(\Gamma)$ sending A_i^* to A_i $(0 \leq i \leq d)$. Furthermore A and A^* have the same characteristic polynomial.*

Proof. The first statement is immediate from the fact that (2.1) and (2.2) apply to both Γ and Γ^*. As a consequence A and A^* have the same minimal polynomial and the same eigenvalues.

From Biggs [2, p. 143] the eigenvalue multiplicities can be computed from the intersection numbers and hence are the same for Γ and Γ^*, proving the second statement. □

Let E be the dimension $r-1$ subspace of $\mathbb{R}^r$ orthogonal to δ_r, and let Φ denote the root system A_{r-1} as given in (1.8). We represent any vertex p of Γ^* (which we recall is a subset of Ω_r) by the vector

$$\hat{p} = \Big(\sum_{i \in p} e_i\Big) - \frac{d}{r}\,\delta_r \tag{2.3}$$

in E. Set

$$\hat{\Gamma}^* = \{\hat{p} \mid p \in \Gamma^*\}. \tag{2.4}$$

There are four things to notice about $\hat{\Gamma}^*$. First, the inner product between two vectors $\hat{p}$ and $\hat{q}$ in $\hat{\Gamma}^*$ depends only on $\partial(p, q)$. In fact one readily computes

$$\langle \hat{p}, \hat{q} \rangle = \frac{d(r-d)}{r} - \partial(p, q). \tag{2.5}$$

Secondly, for any vertex p in Γ^* we have

$$\sum_{\substack{q\\(p,q)\in E\Gamma^*}} \hat{q} = (dr - d^2 - r)\hat{p}. \tag{2.6}$$

Thirdly, for any adjacent vertices p and q in Γ^* we have $\hat{p} - \hat{q} = e_i - e_j$ for some $i, j \in \Omega_r$. In particular

$$\hat{p} - \hat{q} \in \Phi \text{ if and only if } (p, q) \in E\Gamma^*. \tag{2.7}$$

Lastly, one verifies from (2.3) that

$$\langle \hat{p}, s \rangle \quad \text{is an integer for all } p \in \Gamma^*, s \in \Phi. \tag{2.8}$$

Definition 2.2. Let E be a Euclidean space of dimension $r-1$. A *representation* of Γ in E is a pair σ, Φ, consisting of a function $\sigma: \Gamma \to E$ from the vertex set to E, and a simply laced root system Φ, such that:

(i) $\langle \sigma(u), \sigma(v) \rangle = d(r-d)/r - \partial(u, v) \quad u, v \in \Gamma;$ (2.9)

(ii) $\sum_{(u,v)\in E\Gamma} \sigma(v) = (dr - d^2 - r)\sigma(u)$ for all $u \in \Gamma$; (2.10)

(iii) Φ and $\sigma(\Gamma)$ each span E; (2.11)

(iv) $\sigma(u) - \sigma(v) \in \Phi$ if and only if $(u, v) \in E\Gamma \quad u, v \in \Gamma$; (2.12)

(v) $\langle \sigma(u), s \rangle$ is an integer for all $u \in \Gamma$ and $s \in \Phi$. (2.13)

We show in Theorem 2.4 that Γ has a representation. We first need a preliminary result.

Lemma 2.3. *There is a Euclidean space E of dimension $r-1$ and a function $\sigma: \Gamma \to E$ satisfying* (2.9) *and* (2.10).

Proof. We exploit the algebraic properties A and A^* share. Let E_1^* be the Gram matrix of $\hat{\Gamma}^*$. From (2.5) we see that

$$E_1^* = \sum_{i=0}^{d} \left(\frac{d(r-d)}{r} - i \right) A_i^*. \tag{2.14}$$

It is immediate from (2.6) that for all $u, v \in \Gamma^*$ we have

$$\sum_{\substack{w\\(w,v)\in E\Gamma^*}} \langle \hat{w}, \hat{u} \rangle = (dr - d^2 - r)\langle \hat{u}, \hat{v} \rangle.$$

This implies

$$E_1^* A^* = (dr - d^2 - r)E_1^*, \tag{2.15}$$

as can be seen by comparing entries on both sides.

Now set

$$E_1 = \sum_{i=0}^{d} \left(\frac{d(r-d)}{r} - i\right) A_i. \tag{2.16}$$

By Lemma 2.1, we must also have

$$E_1 A = (dr - d^2 - r)E_1. \tag{2.17}$$

The Gram matrix E_1^* is positive semi-definite with rank $r-1$. By Lemma 2.1 and the remark proceeding it, E_1 and E_1^* share the same characteristic polynomial, so this is also true of E_1. We can now interpret E_1 as a Gram matrix for a system of vectors $\{\sigma(u) \mid u \in \Gamma\}$ spanning some $r-1$ dimensional Euclidean space E. (2.9) and (2.10) now follow from (2.16) and (2.17), respectively. □

Theorem 2.4. *Γ has a representation in some Euclidean space of dimension $r-1$.*

Proof. Let σ and E be as given in Lemma 2.3. We need to produce a root system Φ in E satisfying (2.11)–(2.13). Set

$$\Delta = \{\sigma(u) - \sigma(v) \mid (u, v) \in E\Gamma,\ u, v \in \Gamma\}. \tag{2.18}$$

We claim Δ spans E. To see this, note that because of (2.10) and the fact that $b_0 = d(r-d)$, for any vertex $u \in \Gamma$ we have

$$\sum_{\substack{v \\ (u,v)\in E\Gamma}} (\sigma(u) - \sigma(v)) = (dr - d^2)\sigma(u) - (dr - d^2 - r)\sigma(u),$$

giving

$$\sigma(u) = \frac{1}{r}\left(\sum_{(u,v)\in E\Gamma} (\sigma(u) - \sigma(v))\right). \tag{2.19}$$

Thus $\sigma(\Gamma)$ is in the subspace of E spanned by Δ. Since $\sigma(\Gamma)$ spans E this proves the claim.

We now show Δ satisfies the conditions of Lemma 1.8. (i) is satisfied for if $(u, v) \in E\Gamma$, by (2.9) we have

$$\begin{aligned}\langle \sigma(u) - \sigma(v), \sigma(u) - \sigma(v)\rangle &= 2\left\{\frac{d(r-d)}{r} - \left(\frac{d(r-d)}{r} - 1\right)\right\} \\ &= 2. \end{aligned} \tag{2.20}$$

Also (ii) is satisfied, for if $s = \sigma(u) - \sigma(v)$ and $t = \sigma(w) - \sigma(z)$ $(u, v, w, z \in \Gamma)$ are any two vectors in Δ, eq. (2.9) implies

$$\langle s, t\rangle = \partial(v, w) - \partial(u, w) + \partial(u, z) - \partial(v, z),$$

an integer. By the Cauchy-Schwartz inequality we have

$$\langle s, t\rangle^2 \leqslant \langle s, s\rangle\langle t, t\rangle = 4$$

so

$$\langle s, t\rangle \in \{0, \mp 1, \mp 2\}.$$

Lemma 1.8 now says Δ is contained in a root system Φ, which clearly spans E since Δ does. Thus (2.11) holds. Now (2.12) holds because of how Δ was constructed and the fact that if $s=\sigma(u)-\sigma(v)$ is in Φ for some $u, v \in \Gamma$, then $\langle s, s\rangle = 2$, forcing

$$\langle \sigma(u), \sigma(v)\rangle = d(r-d)/r - 1$$

and putting (u, v) in $E\Gamma$.

To prove Φ satisfies (2.13) let u be any vertex in Γ and let s be any root in Φ. From the proof of Lemma 1.8, s is an integral linear combination of roots in Δ, so it suffices to show (2.13) holds if $s \in \Delta$. In this case $s = \sigma(w) - \sigma(z)$ for some adjacent vertices w and z in Γ, so

$$\begin{aligned}\langle \sigma(u), s\rangle &= \langle \sigma(u), \sigma(w)\rangle - \langle \sigma(u), \sigma(z)\rangle \\ &= \partial(u, z) - \partial(u, w),\end{aligned}$$

an integer. □

For the rest of this paper σ, E, and Φ will refer to the representation of Γ in Theorem 2.4. We show Φ is irreducible, so by Lemma 1.7, for $r \neq 7, 8, 9$, Φ is either A_{r-1} or D_{r-1}. We then show in Theorem 2.14 that $\Phi = D_{r-1}$ could not occur, and in Theorem 2.15 that if $\Phi = A_{r-1}$ we must have $\Gamma \simeq \Gamma^*$.

We first make a few remarks and definitions.

Lemma 2.5. *Suppose $\{u, v, w\}$ ($u, v, w \in \Gamma$) is a walk in Γ with $\partial(u, w) = 2$, and suppose there exists a vertex z adjacent to u and w but not v. Then*

$$\sigma(z) = \sigma(u) + \sigma(w) - \sigma(v). \tag{2.21}$$

In particular, z is unique if it exists.

Proof. Set $p = \sigma(z) - \sigma(u) + \sigma(v) - \sigma(w)$. From (2.9) we get $\langle p, p\rangle = 0$ so p is the zero vector, yielding (2.21). □

Definition 2.6. For any vertex $u \in \Gamma$ let

$$\Phi_u = \{\sigma(u) - \sigma(v) \mid v \in \Gamma, (u, v) \in E\Gamma\}.$$

Φ_u is the set of roots in Φ representing edges in $E\Gamma$ containing u.

Lemma 2.7. *For any $u \in \Gamma$ and any distinct vertices $v, w \in \Gamma$ adjacent to u, the roots $\sigma(u) - \sigma(v)$ and $\sigma(u) - \sigma(w)$ in Φ_u satisfy*

$$\langle \sigma(u) - \sigma(v), \sigma(u) - \sigma(w)\rangle = \begin{cases} 1 & \text{if } \partial(v, w) = 1, \\ 0 & \text{if } \partial(v, w) = 2. \end{cases}$$

Proof. From (2.9),

$$\begin{aligned}\langle \sigma(u)-\sigma(v), \sigma(u)-\sigma(w)\rangle &= \partial(u, w)+\partial(v, u)-\partial(v, w)\\ &= 2-\partial(v, w).\end{aligned}$$

This proves the lemma. □

Lemma 2.8. *Let u be any vertex in Γ. Then*

$$\sigma(u)=\frac{1}{r}\left(\sum_{s\in\Phi_u} s\right).$$

Proof. This is immediate from (2.19). □

Lemma 2.9. *Φ is irreducible.*

Proof. By Lemma 1.8, this is false if and only if the set Δ in (2.18) is an orthogonal union of some non-empty subsets Δ_1 and Δ_2. In this case, the connectivity of Γ implies there is a vertex u in Γ adjacent to vertices v and w with $\sigma(u)-\sigma(v)\in\Delta_1$ and $\sigma(u)-\sigma(w)\in\Delta_2$. By Lemma 2.7 we have $\partial(v, w)=2$. Since $c_2=4$, there are three vertices in Γ besides u that are adjacent to v and w. By Lemma 2.5 at least two of these three are adjacent to u. Denoting either of these vertices by z, we see by Lemma 2.7 that $\sigma(u)-\sigma(z)$ has inner product 1 with both $\sigma(u)-\sigma(v)$ and $\sigma(u)-\sigma(w)$ so it is not in Δ_1 or Δ_2, contradicting our opening remarks. □

Definition 2.10. Let $\alpha\in\{1, -1\}$. We say two roots in D_n are *paired* if they are of the form $e_i+\alpha e_j$ and $-e_i+\alpha e_j$ for some integers $i, j\in\Omega_n$.

We note that if roots $s, t\in D_n$ are paired then $\langle s, t\rangle=0$ and for all roots $p\in D_n\backslash\{\pm s, \pm t\}$,

$$|\langle p, s\rangle| = |\langle p, t\rangle|, \tag{2.22}$$

where $|\ |$ denotes absolute value.

Lemma 2.11. *If $\Phi=D_{r-1}$, then Φ_u does not contain paired roots for any vertex $u\in\Gamma$.*

Proof. Fix $u\in\Gamma$ and suppose $v, w\in\Gamma$ are vertices adjacent to u where $\sigma(u)-\sigma(v)$ and $\sigma(u)-\sigma(w)$ are paired. By Lemma 2.7, we have $\partial(v, w)=2$. Now the intersection number $a_2=0$, for otherwise let $x\in\Gamma$ satisfy

$$\partial(x, w)+1=\partial(x, v)=2,$$

and set $q = \partial(x, u)$ ($q = 1$ or 2). Then by (2.9),

$$\begin{aligned}\langle \sigma(x) - \sigma(w), \sigma(u) - \sigma(v) \rangle &= \partial(x, v) + \partial(w, u) - \partial(x, u) - \partial(w, v) \\ &= 1 - q \end{aligned} \tag{2.23}$$

and

$$\begin{aligned}\langle \sigma(x) - \sigma(w), \sigma(u) - \sigma(w) \rangle &= \partial(x, w) + \partial(u, w) - \partial(x, u) \\ &= 2 - q. \end{aligned} \tag{2.24}$$

interpreting s, t, and p in (2.22) as $\sigma(u) - \sigma(v)$, $\sigma(u) - \sigma(w)$, and $\sigma(x) - \sigma(w)$, (2.23) and (2.24) imply $|\langle p, s \rangle| \neq |\langle p, t \rangle|$, a contradiction. □

Definition 2.12. If $\Phi = D_{r-1}$, we can assume $E = \mathbb{R}^{r-1}$ and for all vertices $u \in \Gamma$ we write

$$\sigma(u) = \sum_{i=1}^{r-1} a_u(i) e_i, \quad e_i \in B^{r-1}, i \in \Omega_{r-1}.$$

If $\Phi = A_{r-1}$, we can assume $E = \delta_r^{\perp}$ in $\mathbb{R}^r$, and for all $u \in \Gamma$ write

$$\sigma(u) = \sum_{i=1}^{r} a_u(i) e_i, \quad e_i \in B^r, i \in \Omega_r,$$

where it is understood

$$\sum_{i=1}^{r} a_u(i) = 0. \tag{2.25}$$

Lemma 2.13. *Suppose $\Phi = A_{r-1}$ or D_{r-1}, and set $t = 0$ or 1 depending on whether $\Phi = A_{r-1}$ or not. Using the notation of Definition 2.12 we have*

$$|a_u(i)| < 1, \quad u \in \Gamma, i \in \Omega_{r-t}. \tag{2.26}$$

Proof. For any vertex $u \in \Gamma$, the subgraph of Γ induced on the set of vertices in Γ adjacent to u is regular with valency $a_1 = r - 2$. By Lemma 2.7, the vertices in this subgraph are associated with roots in Φ_u, two roots having inner product 1 if and only if the vertices they represent are adjacent. Hence each root in Φ_u has inner product 1 with exactly $r - 2$ other roots in Φ_u. For all $i \in \Omega_{r-t}$ let $k_u^+(i)$ be the number of roots in Φ_u of the form $e_i \mp e_j$ for some $j \in \Omega_{r-t}$, and let $k_u^-(i)$ be the number of roots in Σ_u of the form $-e_i \mp e_j$ for some $j \in \Omega_{r-t}$. By Lemma 2.8 we have

$$a_u(i) = \frac{1}{r} \{k_u^+(i) - k_u^-(i)\}. \tag{2.27}$$

Of course, by Lemma 2.11 at most one of $k_u^+(i)$ and $k_u^-(i)$ is non-zero. We conclude (2.26) holds, since otherwise $k_u^+(i)$ or $k_u^-(i)$ is at least r, forcing (by Lemma 2.11) any root in Φ_u involving i to have inner product 1 with at least $r - 1$ other roots in Φ_u, contradicting our remarks above. □

Theorem 2.14. *Φ is not of type D_{r-1}.*

Proof. Assume on the contrary that $\Phi = D_{r-1}$. By Theorem 2.4 and (2.13)

$$\langle \sigma(u), s\rangle \in \mathbb{Z}, \quad u \in \Gamma, s \in \Phi$$

so

$$xa_u(i) + ya_u(j) \in \mathbb{Z}, \quad u \in \Gamma, i, j, \in \Omega_{r-1}, x, y \in \{1, -1\}.$$

This means that for any $u \in \Gamma$, either $a_u(i)$ is an integer for all $i \in \Omega_{r-1}$ or $a_u(i) + \frac{1}{2}$ is an integer for all $i \in \Omega_{r-1}$. By Lemma 2.13, the first possibility implies $\sigma(u) = 0$, contradicting (2.9) (with $u = v$), so it does not occur, and the second possibility implies $a_u(i) = \mp\frac{1}{2}$ for all $i \in \Omega_{r-1}$. We conclude $\sigma(u)$ is of the form

$$\sigma(u) = \tfrac{1}{2}\Big(\sum_{i=1}^{r-1} \mp e_i\Big), \quad u \in \Gamma. \tag{2.28}$$

Computing $\langle \sigma(u), \sigma(u)\rangle$ for any vertex $u \in \Gamma$ in two ways using (2.9) and (2.28) we obtain

$$\frac{r-1}{4} = \frac{d(r-d)}{r} \tag{2.29}$$

or

$$d = \frac{r - \sqrt{r}}{2}. \tag{2.30}$$

Setting $r = t^2$ for some integer t ($3 \leq t$) in (2.30) we get $d = \binom{t}{2}$. Now the three sets

$$u^* = \left\{1, 2, \ldots, \binom{t}{2}\right\},$$

$$v^* = \left\{t+1, t+2, \ldots, \binom{t+1}{2}\right\},$$

$$w^* = \left\{\binom{t+1}{2} + 1, \ldots, t^2\right\},$$

thought of as vertices in Γ^*, satisfy

$$\partial(u^*, v^*) = t, \tag{2.31}$$

$$\partial(u^*, w^*) = d, \tag{2.32}$$

$$\partial(v^*, w^*) = d. \tag{2.33}$$

This means the intersection number s_{tdd} for Γ^* and hence Γ, is positive, so there exist three vertices u, v, and w in Γ satisfying (2.31)–(2.33). We show this is inconsistent with (2.28).

Let $P_1, P_2, P_3 \subseteq \Omega_{r-1}$ be the set of indices for which the components of $\sigma(u)$ and $\sigma(v)$, $\sigma(v)$ and $\sigma(w)$, and $\sigma(u)$ and $\sigma(w)$ in (2.28) differ, respectively, and set

$$x = |P_1 \cap P_2|, \qquad y = |P_2 \cap P_3|, \qquad z = |P_1 \cap P_3|,$$
$$\alpha = |\Omega_{r-1} \backslash \{P_2 \cup P_3\}|.$$

By (2.28), all components of $\sigma(u)$, $\sigma(v)$, and $\sigma(w)$ are either $\frac{1}{2}$ or $-\frac{1}{2}$, so $P_1 \subseteq P_2 \cup P_3$ and $P_1 \cap P_2 \cap P_3 = \emptyset$. In particular $P_1 = (P_1 \cap P_2) \cup (P_1 \cap P_3)$ and thus $|P_1| = x + z$. Similarly $|P_2|$ and $|P_3|$ are $x + y$ and $y + z$, respectively, and

$$\alpha = r - 1 - x - y - z. \tag{2.34}$$

From (2.9), (2.28), (2.29) and the definition of P_1 we get

$$\begin{aligned}\langle \sigma(u), \sigma(v) \rangle &= \frac{r-1}{4} - t \\ &= \tfrac{1}{4}(r - 1 - 2\,|P_1|).\end{aligned}$$

We conclude

$$x + z = 2t. \tag{2.35}$$

Considering $\langle \sigma(v), \sigma(w) \rangle$ and $\langle \sigma(u), \sigma(w) \rangle$ we also get

$$x + y = r - t \tag{2.36}$$

and

$$y + z = r - t. \tag{2.37}$$

But now solving for x, y, and z in (2.35)–(2.37) and then for α in (2.34) we get $\alpha = -1$, an impossibility. This proves the theorem. □

Theorem 2.15. *If Φ is of type A_{r-1} then $\Gamma \simeq \Gamma^*$.*

Proof. Assume $\Phi = A_{r-1}$. From Theorem 2.4 and (2.13) we have

$$\langle \sigma(u), s \rangle \in \mathbb{Z}, \quad u \in \Gamma,\ s \in \Phi$$

so

$$a_u(i) - a_u(j) \in \mathbb{Z}, \quad u \in \Gamma,\ i, j \in \Omega_r.$$

This and Lemma 2.13 imply that for any $u \in \Gamma$, $a_u(i)$ takes on at most 2 distinct values α and $\alpha - 1$ for some real number α $(0 < \alpha < 1)$, as i ranges over Ω_r. The constant α does not depend on $u \in \Gamma$ since for adjacent vertices $v, w \in \Gamma$,

$$\sigma(v) - \sigma(w) = e_i - e_j \quad \text{for some } i, j \in \Omega_r,$$

telling us that for all $h \in \Omega_r$, $a_v(h)$ and $a_w(h)$ differ by an integer. Now let $u \in \Gamma$ be fixed and let $f = f(u)$ $(0 \leq f \leq r)$ be the number of integers $i \in \Omega_r$ where $a_u(i) = \alpha$. By (2.25) we have

$$f\alpha + (r - f)(\alpha - 1) = 0 \quad \text{or} \quad \alpha = (r - f)/r.$$

In particular f does not depend on $u \in \Gamma$, and $f \neq 0$ or r by (2.9).

Replacing vectors in $\sigma(\Gamma)$ by their negatives if necessary, we can assume $f \leq n/2$. If we set

$$P(u) = \{i \mid a_u(i) = \alpha\}, \quad u \in \Gamma,$$

we get

$$\sigma(u)=\Big(\sum_{i\in P(u)} e_i\Big)-\frac{f}{r}\delta_r, \quad u\in\Gamma.$$

Since (2.9) implies

$$\langle\sigma(u),\sigma(u)\rangle=\frac{d(r-d)}{r}$$

we conclude $f=d$.

Now $\sigma(\Gamma)$ is seen to be a subset of (2.3). Since Γ^* and Γ have the same size $\sigma(\Gamma)$ must equal the set in (2.3). The map $u\to P(u)$ is by (2.12) the desired graph isomorphism between Γ and Γ^*. □

We have now shown Theorem 1.1 holds if $r\neq 7, 8, 9$. We now consider the one remaining case.

Lemma 2.16. *Theorem* 1.1 *holds if* $(d, r)=(4, 8)$.

Proof. Here Φ spans a Euclidean space of dimension 7, so in view of Theorem 2.14 and Theorem 2.15 we assume Φ is of type E_7. Recall from (2.18) that Δ is the set of roots in Φ representing (directed) edges in Γ. From (2.9) the vectors in $\sigma(\Gamma)$ have square length 2. From (2.9) and (2.13) and the argument about the Cauchy inequality following (2.20), we see the set $\Delta\cup\sigma(\Gamma)$ satisfies the assumptions of Lemma 1.8, and so is contained in a (irreducible) root system Φ' containing Φ.

Since E_7 is larger than A_7 or D_7, we must have $\Phi'=\Phi$. We also notice

$$\sigma(\Gamma)\cap\Delta=\emptyset, \tag{2.38}$$

for if $\sigma(u)=\sigma(v)-\sigma(w)$ for some vertices $u, v, w\in\Gamma$ with $(v, w)\in E\Gamma$, by (2.9) we have

$$2=\langle\sigma(u),\sigma(u)\rangle=\langle\sigma(u),\sigma(v)-\sigma(w)\rangle=\partial(u,w)-\partial(u,v)$$

contradicting the fact that v and w are adjacent. We will need the following fact: by an examination of the representation (1.11) of E_7 in Example 1.6 one verifies that for any root s in E_7, we have

$$|\{t\mid\langle t,s\rangle=1,\ t\in E_7\}|=32. \tag{2.39}$$

We now claim Δ itself is a simply laced root system.

To prove this, we need to verify (iv) in Definition 1.4. Since $s\in\Delta$ implies $-s\in\Delta$, and since $|\langle s,t\rangle|=2$ implies $s=t$ or $s=-t$, it suffices to show that for any $s, t\in\Delta$ with $\langle s,t\rangle=1$ we have $s-t\in\Delta$. Let $s, t\in\Delta$ with $\langle s,t\rangle=1$ be given and suppose $u, v\in\Delta$ are adjacent vertices satisfying $s=\sigma(u)-\sigma(v)$. Using (1.1)–(1.3) we see there are 20 vertices $w\in\Gamma$ with $\partial(w,u)+1=\partial(w,v)$, so by (2.9), there are precisely 20 vectors $x\in\sigma(\Gamma)$ with $\langle x,s\rangle=1$. Let $\{v_1,\ldots,v_6\}$ be the $a_1 = 6$ vertices in Γ adjacent to both u and v. Then the 12 distinct vectors

$$\{\sigma(u) - \sigma(v_i) \mid 1 \leqslant i \leqslant 6\} \cup \{\sigma(v_i) - \sigma(v) \mid 1 \leqslant i \leqslant 6\} \tag{2.40}$$

all have inner product 1 with s. By (2.39) and the remarks above, we see t must be a vector in (2.40).

However, if $t = \sigma(u) - \sigma(v_i)$ or $\sigma(v_i) - \sigma(v)$ for some integer i ($1 \leqslant i \leqslant 6$), we have either

$$\begin{aligned} s - t &= \sigma(u) - \sigma(v) - \sigma(u) + \sigma(v_i) \\ &= \sigma(v_i) - \sigma(v) \end{aligned}$$

or

$$\begin{aligned} s - t &= \sigma(u) - \sigma(v) - \sigma(v_i) + \sigma(v) \\ &= \sigma(u) - \sigma(v_i), \end{aligned}$$

and so in either case

$$s - t \in \Delta.$$

Thus Δ is a simply laced root system. By (2.38) Δ is now a proper subset of Φ, so Δ must be of type A_7 or D_7. But then σ, Δ is a representation of Γ to which Theorem 2.14 or Theorem 2.15 applies, forcing $\Delta = A_7$ and $\Gamma \simeq \Gamma^*$. □

We note that Corollary 1.2 was obtained independently by Neumaier as part of a classification of distance-regular graphs satisfying $b_i = (d - i)(a - ci)/2$ and $c_i = i + c\binom{i}{2}$ ($1 \leqslant i \leqslant d$) for some integers a and c.

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